

A lower bound for the mass of axisymmetric connected black hole data sets

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Abstract

We present a generalisation of the Brill-type proof of positivity of mass for axisymmetric initial data to initial data sets with black hole boundaries. The argument leads to a strictly positive lower bound for the mass of simply connected, connected axisymmetric black hole data sets in terms of the mass of a reference Schwarzschild metric.

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1 Introduction

In [6], the first author extended the validity of the axisymmetric positive mass theorems of Brill [4], Moncrief (unpublished), Dain (unpublished) and Gibbons and Holzegel [9] to all asymptotically flat initial data on \mathbb{R}^3 invariant under a $U(1)$ -action with positive Ricci scalar. The object of this work is to show how to adapt the analysis to the case where black hole boundaries are present in the initial data. This leads to a strictly positive lower bound for the mass for initial data sets containing a connected “non-degenerate horizon”.

Let (M, g) be a simply connected three dimensional Riemannian manifold with boundary ∂M which admits a Killing vector field with periodic orbits and is the union of a compact set and of one asymptotically flat end.

By [6], g admits a global coordinate system in which the metric takes the form

$$ds^2 = e^{-2U_\# + 2\alpha_\#} (d\rho_\#^2 + dz_\#^2) + \rho_\#^2 e^{-2U_\#} (d\varphi + \rho_\# \bar{B} d\rho_\# + \bar{A} dz_\#)^2, \quad (1.1)$$

where ∂_φ is the Killing vector field, $\varphi \in [0, 2\pi)$, the coordinates $(\rho_\#, z_\#)$ cover $([0, \infty) \times \mathbb{R}) \setminus \mathring{K}$ for some compact set K whose intersection with the axis $\{\rho_\# = 0\}$ is connected and non-empty. Here \mathring{K} denotes the interior of K . The choice of K is never unique. However, we show in the present paper that it cannot be arbitrary either. In fact, if one lets K' be the compact set obtained by adjoining to K its reflection about the $\rho_\#$ axis in the $(\rho_\#, z_\#)$ -plane, then the logarithmic capacity of K' (with respect to the $(\rho_\#, z_\#)$ -plane) depends uniquely on the geometry of (M, g) .

The above property of K leads to two canonical choices of K : one can use either a line segment of length $2m_1$ on the $\rho_\#$ -axis, or a half-disc of radius $\frac{m_1}{2}$ centered on the axis, where m_1 is twice the logarithmic capacity of K' . In the stationary vacuum case, those coordinate systems are respectively known as Weyl coordinates and isotropic (or spherical) coordinates. In the static case, i.e. (M, g) is a Schwarzschild slice, m_1 coincides with the Schwarzschild mass. For the maximal slice of the Kerr metric, $m_1 = \sqrt{m^2 - a^2}$.

The main result of the present paper is as follows.

THEOREM 1.1 *Let (M, g) be a smooth simply connected three-dimensional manifold which has a smooth connected compact boundary ∂M , is asymptotically flat with one end and satisfies (2.1) for some $k \geq 5$ and (2.2). Furthermore, assume that (M, g) admits a Killing vector field with periodic orbits. If M has non-negative scalar curvature and if the mean curvature of*

∂M with respect to the normal pointing towards M is non-positive, then the ADM mass of (M, g) satisfies

$$m > \frac{\pi}{4} m_1, \quad (1.2)$$

where m_1 is the positive constant obtained in Theorems 2.2 and 2.4.

Even though the constant m_1 is uniquely determined, it should be admitted that it is not easy to determine m_1 if the metric is not given directly in the coordinate system (2.15) or (2.27). In the general case, one needs to solve a PDE on M , and then m_1 can be read off from the asymptotic behaviour of the solution at infinity, see Proposition 2.6.

We note that the simple-connectedness of M would be a consequence of the topological censorship theorem of [8] if M were a Cauchy hypersurface for $J^+(M) \cap J^-(\mathcal{I}^+)$. We are grateful to G. Galloway for pointing this out.

A satisfactory generalisation of our result to degenerate horizons would require a thorough understanding of the behaviour of the metric near such horizons, a problem which is widely unexplored. We simply note that positivity of m is easily established by similar methods if one assumes, e.g., that (M, g) has no boundary but contains instead suitably defined asymptotically cylindrical ends. In this case, whenever a twist potential exists one further has the stronger Dain-type inequality controlling the mass from below in terms of a positive quantity, which equals the square root of the length of the angular-momentum for connected configurations.

We conjecture that the sharp inequality is

$$m \geq m_1, \quad (1.3)$$

with equality if and only if M is a time-symmetric Cauchy hypersurface for the d.o.c. of the Schwarzschild-Kruskal-Szekeres space-time.

Ideally, one would like to obtain a simple proof of the Penrose inequality in the current setting, but we have not been able to achieve this. We make some comments about that in the appendix.

We note the recent paper [1], where a lower bound for minimal-surface area in terms of angular-momentum is established under a set of restrictive conditions; see also [11]. Our construction of global coordinates is relevant for the analysis in [1].

2 Axisymmetric black hole data sets

Let (M, g) be a three-dimensional smooth simply connected manifold with a smooth connected compact boundary ∂M . On (M, g) , we assume that there

is a Killing vector field η with periodic orbits, among which the principal ones are assumed, without loss of generality, to have period 2π .

(M, g) will be assumed to have *one* asymptotically flat end in the usual sense that there exists a region $M_{\text{ext}} \subset M$ diffeomorphic to $\mathbb{R}^3 \setminus B_R$, where B_R is a coordinate ball of radius R , such that in local coordinates on M_{ext} obtained from $\mathbb{R}^3 \setminus B_R$ the metric satisfies the fall-off conditions, for some $k \geq 1$,

$$g_{ij} - \delta_{ij} = o_k(r^{-1/2}) , \quad (2.1)$$

$$\partial_k g_{ij} \in L^2(M_{\text{ext}}) , \quad (2.2)$$

$$R^i{}_{jkl} = o(r^{-5/2}) , \quad (2.3)$$

where we write $f = o_k(r^\mu)$ if f satisfies

$$\partial_{k_1 \dots k_l} f = o(r^{\mu-l}) \text{ for } 0 \leq l \leq k . \quad (2.4)$$

The boundary behaviour near the event horizon of all functions of interest has been established in [7] in a closely related context. For completeness, and to make it clear that it applies to our setting, we outline the analysis of [7] in what follows.

2.1 Reduction to the boundaryless case

Since M is simply connected and ∂M is connected, [10, Lemma 4.9] implies that

$$\partial M \text{ has the topology of a 2-sphere,} \quad (2.5)$$

and can be filled by a ball, say $B_\#$. Moreover, the metric g and the $U(1)$ -action induced by η extend to the extended manifold $M_\# := M \cup B_\#$.

Let

$$\mathcal{A} := \{p \in M : g(\eta, \eta)|_p = 0\} , \quad \mathcal{A}_\# := \{p \in M_\# : g(\eta, \eta)|_p = 0\} .$$

Denote by $Q_\# := M_\#/U(1)$, $Q := M/U(1)$ and $\partial M/U(1)$ the collections of the orbits of the group of isometries generated by η on $M_\#$, M and ∂M , respectively. It is known that $Q_\#$ is a manifold with boundary $\mathcal{A}_\#/U(1) \approx \mathcal{A}_\#$, while Q is a manifold with boundary

$$(\mathcal{A}/U(1)) \cup (\partial M/U(1)) \approx \mathcal{A} \cup (\partial M/U(1)) .$$

As ∂M is a sphere and invariant under $U(1)$, it contains exactly two fixed points, say p_n and p_s , of $U(1)$. In other words,

$$\partial M \cap \mathcal{A} = \{p_n, p_s\} .$$

It is readily seen that $\partial M/U(1)$ is a smooth curve in $Q_\#$ with endpoints p_n and p_s . Additionally, at both p_n and p_s , $\partial M/U(1)$ intersects $\mathcal{A}_\#$ at a right angle.

In the sequel, we will assume that (2.1) holds for some $k \geq 5$. Note that this implies (2.3). By [6, Theorem 2.7], the metric g on $M_\#$ admits the following representation

$$g = e^{-2U_\# + 2\alpha_\#} (d\rho_\#^2 + dz_\#^2) + \rho_\#^2 e^{-2U_\#} (d\varphi + \rho_\# \bar{B} d\rho_\# + \bar{A} dz_\#)^2, \quad (2.6)$$

where ∂_φ is the rotational Killing vector field, and $M_\#$ can be identified with \mathbb{R}^3 on which $(\rho_\#, z_\#, \varphi)$ are its cylindrical coordinates. Furthermore, $U_\#, \alpha_\#, \bar{B}$ and \bar{A} are smooth functions on $M_\#$ which are φ -independent and satisfy $\alpha_\# = 0$ whenever $\rho_\# = 0$ and

$$\begin{aligned} U_\# &= o_{k-3}(r_\#^{-1/2}), & \alpha_\# &= o_{k-4}(r_\#^{-1/2}), \\ \bar{B} &= o_{k-3}(r_\#^{-5/2}), & \bar{A} &= o_{k-3}(r_\#^{-3/2}) \text{ for } r_\# = \sqrt{\rho_\#^2 + z_\#^2} \rightarrow \infty. \end{aligned} \quad (2.7)$$

It is useful to consider the manifolds $Q_\#^{(2)}$ and $Q^{(2)}$ obtained by doubling $Q_\#$ and Q along $\mathcal{A}_\#/U(1)$ and $\mathcal{A}/U(1)$, respectively. Naturally, $Q^{(2)}$ injects into $Q_\#^{(2)}$. In addition, $\rho_\#$ and $z_\#$ can be extended naturally to $Q_\#^{(2)}$ so as to make $\rho_\#$ an odd function about $\mathcal{A}_\#$ while $z_\#$ an even function. Then $Q_\#^{(2)}$ can be identified with the complex plane $\mathbb{C}_\# := \{\zeta_\# := \rho_\# + iz_\#\}$ in which

$$Q_\# \approx \{\zeta_\# : \operatorname{Re} \zeta_\# \geq 0\}, \quad \mathcal{A}_\# \approx \{\zeta_\# : \operatorname{Re} \zeta_\# = 0\},$$

where \approx denotes “diffeomorphic to”. This implies in particular that $Q_\#$ has a natural complex structure. Furthermore, in this picture, $Q^{(2)}$ is an unbounded (open) subset, denoted by $\Omega_\#$, of $\mathbb{C}_\#$ whose boundary is a smooth connected closed curved,

$$Q \approx \Omega_\# \cap \{\zeta_\# : \operatorname{Re} \zeta_\# \geq 0\} \text{ and } \mathcal{A} \approx \Omega_\# \cap \{\zeta_\# : \operatorname{Re} \zeta_\# = 0\}.$$

2.2 Pseudo-spherical coordinates

We proceed to modify $(\rho_\#, z_\#, \varphi)$ to a coordinate system (ρ_S, z_S, φ) on M such that ∂M corresponds to a sphere $\{\rho_S^2 + z_S^2 = \text{const}\}$. An approach to achieve this is to follow the procedure in [7] to first construct Weyl coordinate functions and then transform them to the desired form. We present here a simpler approach, directly tied to the theory of conformal mappings. As

will be seen, this also provides an alternative to the construction of Weyl coordinates in [7].

Without loss of generality, we assume that $\Omega_\#$ does not contain the origin of $\mathbb{C}_\#$. Let Θ denote the inversion map of $\mathbb{C}_\#$ about the unit circle $\partial D_\#(0, 1)$ and define $G_\# = \Theta(\Omega_\#) \cup \{0\}$. Note that as $\partial\Omega_\#$ is a smooth simple closed curved, so is $\partial G_\#$. This implies that $G_\#$ is simply connected. Let h_1 be the solution to the problem

$$\begin{cases} \Delta^\# h_1 = 0 & \text{in } G_\# , \\ h_1 = -\frac{1}{2} \log(\rho_\#^2 + z_\#^2) & \text{on } \partial G_\# , \end{cases}$$

where $\Delta^\#$ is the Laplace operator of $d\rho_\#^2 + dz_\#^2$, and h_2 be a harmonic conjugate of h_1 , i.e. h_2 satisfies

$$\partial_{\rho_\#} h_1 = \partial_{z_\#} h_2 , \quad \partial_{z_\#} h_1 = -\partial_{\rho_\#} h_2 .$$

Define

$$\Psi \equiv \psi_1 + i\psi_2 := \zeta_\# \exp(h_1 + i h_2) ;$$

recall that $\zeta_\# = \rho_\# + iz_\#$. Evidently, Ψ is holomorphic, fixes the origin, and maps $\partial G_\#$ to the unit circle $\partial D_\#(0, 1)$. Furthermore, by the definitions of h_1, h_2 and standard elliptic theory, $\Psi \in C^\infty(\bar{G}_\#)$.

We claim that Ψ is “the” Riemann map which maps $G_\#$ one-to-one and onto the unit disc $D_\#(0, 1)$. Indeed, let $\tilde{\Psi}$ be a Riemann map of $G_\#$ which fixes the origin. Then $\tilde{\Psi} = \zeta_\# \tilde{H}$ for some holomorphic function \tilde{H} . Additionally, as $\tilde{\Psi}$ is one-to-one, \tilde{H} is nowhere vanishing. Since $G_\#$ is simply connected, this implies $\tilde{H} = \exp \tilde{h}$ for some holomorphic function \tilde{h} . As $\tilde{\Psi}(\partial G_\#) \subset \partial D_\#(0, 1)$, it follows that

$$\operatorname{Re} \tilde{h} = -\log |\zeta_\#| = -\frac{1}{2} \log(\rho_\#^2 + z_\#^2) \text{ on } \partial G_\# .$$

By uniqueness of solutions of the Laplace equation, we thus have $\operatorname{Re} \tilde{h} \equiv h_1$, which implies $\operatorname{Im} \tilde{h} \equiv h_2 + C$ for some constant C . The claim follows.

As a Riemann map, Ψ has an inverse $\Psi^{-1} : D_\#(0, 1) \rightarrow G_\#$. Since $G_\#$ is a Jordan domain, Ψ^{-1} extends to a homeomorphism of the closed domains thanks to Carathéodory theorem (see e.g. [15, Theorem 14.19]). We claim that this extension is of $C^\infty(\bar{D}_\#(0, 1))$ -differentiability class, and in fact is a diffeomorphism up-to-boundary. By the Inverse Function Theorem, it suffices to show that Ψ' is nowhere vanishing in $\bar{G}_\#$. Furthermore, since Ψ is holomorphic and one-to-one in $G_\#$, it suffices to show that Ψ' does not vanish on $\partial G_\#$. Consider a point $p \in \partial G_\#$ and let $q = \Psi(p) \in \partial D_\#(0, 1)$. Without

loss of generality, we can assume that $q = -i$. Pick a $\delta > 0$ sufficiently small such that

$$\Psi(G_{\#} \cap D_{\#}(p, \delta)) \subset \bar{D}_{\#}(0, 1) \cap D_{\#}(q, \frac{1}{10}) . \quad (2.8)$$

Since $\psi_1^2 + \psi_2^2 = 1$ on $\partial G_{\#}$ we find that, near p , the function

$$F := \psi_2 + \sqrt{1 - \psi_1^2}$$

satisfies

$$\begin{aligned} F &> 0 \text{ in } G_{\#} \cap D_{\#}(p, \delta) , \\ F &= 0 \text{ on } \partial G_{\#} \cap D_{\#}(p, \delta) . \end{aligned}$$

Since ψ_1 and ψ_2 are harmonic, we have

$$\Delta^{\#} F = -\frac{1}{(1 - \psi_1^2)^{3/2}} |\nabla \psi_1|^2 \leq 0 \text{ in } G_{\#} \cap D_{\#}(p, \delta) .$$

It hence follows from the Hopf lemma that

$$\partial_{\nu} \psi_2(p) = \partial_{\nu} F(p) > 0 ,$$

where ∂_{ν} is the derivative in the direction of the inward pointing normal, which gives $|\Psi'(p)| \geq |\partial_{\nu} \psi_2(p)| > 0$. Since p is arbitrary, we thus conclude that Ψ' is always non-zero on $\partial G_{\#}$ and so on $\bar{G}_{\#}$, whence the claim.

Note that we have recovered the Kellogg-Warschawski theorem (see [14, Theorem 3.6] or the original papers [13, 16, 17] of Kellogg and of Warschawski):

PROPOSITION 2.1 *Let $G \subset \mathbb{C}$ be a simply connected bounded domain whose boundary ∂G is $C^{k, \alpha}$ -regular for some $k \geq 2$, $0 < \alpha < 1$, and $\Psi : G \rightarrow D(0, 1)$ its Riemann map. Then Ψ extends to a map in $C^{k, \alpha}(\bar{G})$ and Ψ^{-1} extends to a map in $C^{k, \alpha}(\bar{D}(0, 1))$.*

Define

$$\rho_S + i z_S = \zeta_S := \Theta^{-1} \circ \left(\frac{1}{|\Psi'(0)|} \Psi \right) \circ \Theta . \quad (2.9)$$

Then (ρ_S, z_S) maps $\Omega_{\#}$ one-to-one and onto $\mathbb{C}_S \setminus \bar{D}(0, \frac{m_1}{2})$, where we use the symbol \mathbb{C}_S to denote the complex plane coordinatized by (ρ_S, z_S) , and where

$$m_1 = 2|\Psi'(0)|$$

is twice the logarithmic capacity of $\partial\Omega_\#$. The constant m_1 is related to the Robin constant $\gamma(\partial\Omega_\#)$ of the boundary of $\partial\Omega_\#$ by

$$m_1 = 2 \exp(-\gamma(\partial\Omega_\#)) . \quad (2.10)$$

Note also that by construction, as $\rho_\#^2 + z_\#^2 \rightarrow \infty$, there holds

$$(\rho_S, z_S) - (\rho_\#, z_\#) = O_l((\rho_\#^2 + z_\#^2)^{-1/2}) , \quad l \geq 0 , \quad (2.11)$$

where O_l is defined in a way analogous to (2.4).

We show that m_1 is uniquely determined by (M, g) , i.e. independent of how we form $M_\#$. To see this, let $\tilde{M}_\# = M \cup \tilde{B}_\#$ be a different way of extending M . In $M_\#$ and $\tilde{M}_\#$, the regions representing M are isometric. Hence, if $(\tilde{\rho}_\#, \tilde{z}_\#, \tilde{\varphi})$ is the counterpart in $\tilde{M}_\#$ of $(\rho_\#, z_\#, \varphi)$, then, by (2.6), the map $T : (\rho_\#, z_\#) \mapsto (\tilde{\rho}_\#, \tilde{z}_\#)$ gives a conformal transformation of $Q^{(2)}$. Furthermore, by (2.7), we also have

$$\left| \frac{\partial(\tilde{\rho}_\#, \tilde{z}_\#)}{\partial(\rho_\#, z_\#)} \right| = 1 + O((\rho_\#^2 + z_\#^2)^{-1/2}) \text{ as } \rho_\#^2 + z_\#^2 \rightarrow \infty . \quad (2.12)$$

Hence, if $Q^{(2)}$ is represented by $\tilde{\Omega}_\#$ in the complex plane $\tilde{\mathbb{C}}_\#$ parameterized by $(\tilde{\rho}_\#, \tilde{z}_\#)$, then T defines naturally a bijection of $\Omega_\#$ and $\tilde{\Omega}_\#$, with $T(\infty) = \infty$ and (by (2.12)) $|T'(\infty)| = 1$. It then follows that $\partial\Omega_\#$ and $\partial\tilde{\Omega}_\#$ have the same logarithmic capacity, and so m_1 is independent of the way that the metric has been extended to $B_\#$.

Next, by the uniqueness property of the Laplace equation with Dirichlet boundary data, h_1 is even in the $\rho_\#$ -variable, which implies that, after shifting by a constant, h_2 is odd in the $\rho_\#$ -variable. Using this, one can check that ρ_S is odd while z_S is even in the $\rho_\#$ -variable. In particular, ρ_S vanishes on $\{\rho_\# = 0\} \cap \Omega_\# \approx \mathcal{A}/U(1)$. This implies that ρ_S^2 is a smooth function which vanishes on \mathcal{A} and is even in the $\rho_\#$ -variable. Thus, there is a smooth function χ of $(\rho_\#^2, z_\#)$ such that

$$\rho_S^2 = \rho_\#^2 \chi(\rho_\#^2, z_\#) . \quad (2.13)$$

Furthermore, as

$$(\partial_{\rho_\#} \rho_S)^2 + (\partial_{z_\#} \rho_S)^2 = \left| \frac{\partial(\rho_S, z_S)}{\partial(\rho_\#, z_\#)} \right|$$

is nowhere vanishing in $\overline{\Omega_\#}$ by Proposition 2.1, we also have

$$\chi(\rho_\#^2, z_\#) > 0 \text{ along } \{\rho_\# = 0\} \cap \overline{\Omega_\#} , \text{ and so in } \overline{\Omega_\#} . \quad (2.14)$$

We thus have:

THEOREM 2.2 *Let (M, g) be a three-dimensional smooth simply connected manifold with a smooth connected compact boundary ∂M and assume that (M, g) admits a Killing vector field with periodic orbits. Furthermore, assume that (M, g) has one asymptotically flat end where it satisfies (2.1) for some $k \geq 5$. Then there exists a unique $m_1 > 0$ such that M is diffeomorphic to $\mathbb{R}^3 \setminus B(0, \frac{m_1}{2})$, and, in cylindrical-type coordinates (ρ_S, z_S, φ) on \mathbb{R}^3 , g takes the form*

$$g = e^{-2U_S + 2\alpha_S} (d\rho_S^2 + dz_S^2) + \rho_S^2 e^{-2U_S} (d\varphi + \rho_S \bar{B}_S d\rho_S + \bar{A}_S dz_S)^2, \quad (2.15)$$

where ∂_φ is the rotational Killing vector field, U_S, α_S, \bar{B}_S and \bar{A}_S are smooth functions on M which are φ -independent and satisfy $\alpha_S = 0$ whenever $\rho_S = 0$ and

$$\begin{aligned} U_S &= o_{k-3}(r_S^{-1/2}), & \alpha_S &= o_{k-4}(r_S^{-1/2}), \\ \bar{B}_S &= o_{k-3}(r_S^{-5/2}), & \bar{A}_S &= o_{k-3}(r_S^{-3/2}) \text{ for } r_S = \sqrt{\rho_S^2 + z_S^2} \rightarrow \infty. \end{aligned} \quad (2.16)$$

2.3 Weyl coordinates

We next construct the Weyl coordinates (ρ, z, φ) so that ρ vanishes on both the rotation axis \mathcal{A} and the boundary ∂M . This can be done using a (rotated) Joukovsky transformation,

$$\rho + iz = \zeta := \zeta_S - \frac{m_1^2}{4\zeta_S}. \quad (2.17)$$

Componentwise, we have

$$\rho = \frac{\rho_S [\rho_S^2 + z_S^2 - \frac{m_1^2}{4}]}{\rho_S^2 + z_S^2}, \quad z = \frac{z_S [\rho_S^2 + z_S^2 + \frac{m_1^2}{4}]}{\rho_S^2 + z_S^2}. \quad (2.18)$$

We now check that the map $\zeta_S \mapsto \zeta$ maps $\mathbb{C}_S \setminus \bar{D}(0, \frac{m_1}{2})$ one-to-one and onto $\mathbb{C} \setminus I$ where $I = \{iz : -m_1 \leq z \leq m_1\}$. In view of (2.18), to invert the map it suffices to solve for $|\zeta_S| > \frac{m_1}{2}$. First, note that by (2.17)

$$\zeta \pm im_1 = \frac{1}{\zeta_S} \left(\zeta_S \pm i \frac{m_1}{2} \right)^2. \quad (2.19)$$

It follows that

$$(\zeta + im_1)(\bar{\zeta} + im_1) = \frac{1}{|\zeta_S|^2} \left(|\zeta_S|^2 - \frac{m_1^2}{4} + i \frac{m_1}{2} (\zeta_S + \bar{\zeta}_S) \right)^2.$$

Taking the real part and recalling (2.18) we get

$$\begin{aligned} |\zeta|^2 - m_1^2 &= \frac{1}{|\zeta_S|^2} \left[(|\zeta_S|^2 - \frac{m_1^2}{4})^2 - m_1^2 \rho_S^2 \right] \\ &= \frac{(|\zeta_S|^2 - \frac{m_1^2}{4})^2}{|\zeta_S|^2} - \frac{m_1^2 |\zeta_S|^2}{(|\zeta_S|^2 - \frac{m_1^2}{4})^2} \rho^2. \end{aligned} \quad (2.20)$$

This implies that

$$\frac{(|\zeta_S|^2 - \frac{m_1^2}{4})^2}{|\zeta_S|^2} = \frac{1}{2} \left[|\zeta|^2 - m_1^2 + \sqrt{(|\zeta|^2 - m_1^2)^2 + 4 m_1^2 \rho^2} \right]. \quad (2.21)$$

As $|\zeta_S| > \frac{m_1}{2}$, we thus get

$$\frac{|\zeta_S|^2 - \frac{m_1^2}{4}}{|\zeta_S|} = \frac{1}{\sqrt{2}} \left[|\zeta|^2 - m_1^2 + \sqrt{(|\zeta|^2 - m_1^2)^2 + 4 m_1^2 \rho^2} \right]^{1/2}, \quad (2.22)$$

which implies

$$\begin{aligned} |\zeta_S| &= \frac{1}{2\sqrt{2}} \left\{ \left[|\zeta|^2 - m_1^2 + \sqrt{(|\zeta|^2 - m_1^2)^2 + 4 m_1^2 \rho^2} \right]^{1/2} \right. \\ &\quad \left. + \left[|\zeta|^2 + m_1^2 + \sqrt{(|\zeta|^2 - m_1^2)^2 + 4 m_1^2 \rho^2} \right]^{1/2} \right\}. \end{aligned} \quad (2.23)$$

From what has been said we see that the map $\zeta_S \mapsto \zeta$ maps $\mathbb{C}_S \setminus \bar{D}(0, \frac{m_1}{2})$ one-to-one and onto $\mathbb{C} \setminus I$. In fact, in view of (2.18), (2.21) and (2.22), its inverse is given by

$$\rho_S = (\mu^{1/2} + 1) \frac{\rho}{2}, \quad z_S = (\mu^{-1/2} + 1) \frac{z}{2}, \quad (2.24)$$

where

$$\mu = \frac{|\zeta|^2 + m_1^2 + \sqrt{(|\zeta|^2 - m_1^2)^2 + 4 m_1^2 \rho^2}}{|\zeta|^2 - m_1^2 + \sqrt{(|\zeta|^2 - m_1^2)^2 + 4 m_1^2 \rho^2}}. \quad (2.25)$$

Recall that ρ_S is odd in the ρ_{\sharp} -variable and so vanishes on $\mathcal{A}/U(1)$. Thus, by (2.18), ρ also vanishes on $\mathcal{A}/U(1)$. Also by (2.18), ρ vanishes on $\partial M/U(1)$. Moreover, by (2.17), as $\rho_S^2 + z_S^2 \rightarrow \infty$, there holds

$$(\rho, z) - (\rho_S, z_S) = O_l((\rho_S^2 + z_S^2)^{-1/2}), \quad l \geq 0. \quad (2.26)$$

It thus follows that:

THEOREM 2.3 *The map*

$$(\rho_{\sharp}, z_{\sharp}) \mapsto (\rho, z)$$

defined by (2.9) and (2.17) provides a holomorphic diffeomorphism from \bar{Q} to the complex half-plane $\{\zeta = \rho + iz : \rho > 0\}$.

In the (ρ, z, φ) coordinate system the metric g on M admits again a representation of the form

$$g = e^{-2U+2\alpha}(d\rho^2 + dz^2) + \rho^2 e^{-2U}(d\varphi + \rho B_{\rho} d\rho + A_z dz)^2. \quad (2.27)$$

In the rest of this section, we will use Theorem 2.3 to study the regularity properties of the functions involved.

First, using $g_{\varphi\varphi} = \rho^2 e^{-2U(\rho, z)} = \rho_S^2 e^{-2U_S(\rho_S, z_S)}$ and (2.18), the function U is given by

$$U(\rho, z) := U_S(\rho_S, z_S) - \log \frac{\rho_S}{\rho} = U_S(\rho_S, z_S) - \log \frac{\rho_S^2 + z_S^2}{\rho_S^2 + z_S^2 - \frac{m_1^2}{4}}. \quad (2.28)$$

Recalling (2.22), the above relation can be rewritten as

$$U(\rho, z) = \tilde{U}_S(\rho_S, z_S) + \frac{1}{2} \log \left[\rho^2 + z^2 - m_1^2 + \sqrt{(\rho^2 + z^2 - m_1^2)^2 + 4 m_1^2 \rho^2} \right] \quad (2.29)$$

for some $\tilde{U}_S \in C^\infty(\mathbb{C}_S \setminus D(0, \frac{m_1}{2}))$.

Next, by (2.17),

$$d\rho^2 + dz^2 = \frac{|\zeta_S^2 + \frac{m_1^2}{4}|^2}{|\zeta_S|^4} (d\rho_S^2 + dz_S^2).$$

Thus, using (2.28) and

$$e^{-2U(\rho, z) + 2\alpha(\rho, z)} (d\rho^2 + dz^2) = e^{-2U_S(\rho_S, z_S) + 2\alpha_S(\rho_S, z_S)} (d\rho_S^2 + dz_S^2) \text{ in } \bar{Q}^{(2)}, \quad (2.30)$$

we get

$$\begin{aligned} \alpha(\rho, z) &= U(\rho, z) - U_S(\rho_S, z_S) + \alpha_S(\rho_S, z_S) + \log \frac{|\zeta_S|^2}{|\zeta_S^2 + \frac{m_1^2}{4}|} \\ &= \alpha_S(\rho_S, z_S) + \log \frac{|\zeta_S|^2 - \frac{m_1^2}{4}}{|\zeta_S^2 + \frac{m_1^2}{4}|}. \end{aligned} \quad (2.31)$$

Recalling (2.19), we can rewrite (2.31) as

$$\alpha(\rho, z) = \tilde{\alpha}_S(\rho_S, z_S) + \frac{1}{2} \log \frac{\rho^2 + z^2 - m_1^2 + \sqrt{(\rho^2 + z^2 - m_1^2)^2 + 4m_1^2 \rho^2}}{2\sqrt{\rho^2 + (z - m_1)^2} \sqrt{\rho^2 + (z + m_1)^2}} \quad (2.32)$$

for some $\tilde{\alpha}_S \in C^\infty(\mathbb{C}_S \setminus D(0, \frac{m_1}{2}))$. Also, as α_S vanishes on the axis \mathcal{A} , (2.31) implies that

$$\alpha(\rho, z) = 0 \text{ for } \rho = 0, z \notin [-m_1, m_1]. \quad (2.33)$$

We also need to understand the behaviour of the metric functions B_ρ and A_z , keeping in mind that \bar{B}_S and \bar{A}_S are smooth up-to-boundary. Since

$$\rho_S \bar{B}_S(\rho_S, z_S) d\rho_S + \bar{A}_S(\rho_S, z_S) dz_S = \rho B_\rho(\rho, z) d\rho + A_z(\rho, z) dz,$$

B_ρ and A_z satisfy

$$\begin{aligned} B_\rho(\rho, z) &= \frac{1}{4} [(\mu^{1/2} + 1)^2 + \frac{1}{2}(\mu^{-1/2} + 1) \mu_{,\rho} \rho] \bar{B}_S(\rho_S, z_S) \\ &\quad - \frac{1}{4} \rho^{-1} \mu^{-3/2} \mu_{,\rho} z \bar{A}_S(\rho_S, z_S), \end{aligned} \quad (2.34)$$

$$\begin{aligned} A_z(\rho, z) &= \frac{1}{8} (\mu^{-1/2} + 1) \mu_z \rho^2 \bar{B}_S(\rho_S, z_S) \\ &\quad + \frac{1}{2} [(\mu^{-1/2} + 1) - \frac{1}{2} \mu^{-3/2} \mu_{,z} z] \bar{A}_S(\rho_S, z_S). \end{aligned} \quad (2.35)$$

We compute from (2.25):

$$\begin{aligned} \mu_{,\rho} &= -\frac{4m_1^2 \rho}{\sqrt{(|\zeta|^2 - m_1^2)^2 + 4m_1^2 \rho^2}} \frac{|\zeta|^2 + m_1^2 + \sqrt{(|\zeta|^2 - m_1^2)^2 + 4m_1^2 \rho^2}}{[|\zeta|^2 - m_1^2 + \sqrt{(|\zeta|^2 - m_1^2)^2 + 4m_1^2 \rho^2}]^2}, \\ \mu_{,z} &= -\frac{4m_1^2 z}{\sqrt{(|\zeta|^2 - m_1^2)^2 + 4m_1^2 \rho^2}} \frac{1}{|\zeta|^2 - m_1^2 + \sqrt{(|\zeta|^2 - m_1^2)^2 + 4m_1^2 \rho^2}}. \end{aligned}$$

Also, note that by (2.20) and (2.21),

$$\begin{aligned} |\zeta|^2 - m_1^2 + \sqrt{(|\zeta|^2 - m_1^2)^2 + 4m_1^2 \rho^2} &= \frac{2(|\zeta_S|^2 - \frac{m_1^2}{4})^2}{|\zeta_S|^2}, \\ |\zeta|^2 + m_1^2 + \sqrt{(|\zeta|^2 - m_1^2)^2 + 4m_1^2 \rho^2} &= \frac{2(|\zeta_S|^2 + \frac{m_1^2}{4})^2}{|\zeta_S|^2}, \\ \sqrt{(|\zeta|^2 - m_1^2)^2 + 4m_1^2 \rho^2} &= \frac{(|\zeta_S|^2 - \frac{m_1^2}{4})^2}{|\zeta_S|^2} + \frac{m_1^2 |\zeta_S|^2}{(|\zeta_S|^2 - \frac{m_1^2}{4})^2} \rho^2. \end{aligned}$$

Thus

$$\mu_{,\rho} = -\frac{2m_1^2 |\zeta_S|^4 \rho}{(|\zeta_S|^2 - \frac{m_1^2}{4})^4 + m_1^2 |\zeta_S|^4 \rho^2} \frac{(|\zeta_S|^2 + \frac{m_1^2}{4})^2}{(|\zeta_S|^2 - \frac{m_1^2}{4})^2}, \quad (2.36)$$

$$\mu_{,z} = -\frac{2m_1^2 |\zeta_S|^4 z}{(|\zeta_S|^2 - \frac{m_1^2}{4})^4 + m_1^2 |\zeta_S|^4 \rho^2}. \quad (2.37)$$

We also have, from (2.25) and the before-last displayed equations, that

$$\mu = \left(\frac{|\zeta_S|^2 + \frac{m_1^2}{4}}{|\zeta_S|^2 - \frac{m_1^2}{4}} \right)^2. \quad (2.38)$$

Substituting (2.36), (2.37), (2.38) and (2.24) into (2.34) and (2.35) we obtain

$$\begin{aligned} B_\rho(\rho, z) &= \left[\frac{|\zeta_S|^4}{(|\zeta_S|^2 - \frac{m_1^2}{4})^2} - \frac{m_1^2 |\zeta_S|^2 \rho_S^2 (|\zeta_S|^2 + \frac{m_1^2}{4})}{2(|\zeta_S|^2 - \frac{m_1^2}{4})^2 [(|\zeta_S|^2 - \frac{m_1^2}{4})^2 + m_1^2 \rho_S^2]} \right] \bar{B}_S(\rho_S, z_S) \\ &\quad + \frac{m_1^2 |\zeta_S|^2 z_S}{2(|\zeta_S|^2 - \frac{m_1^2}{4}) [(|\zeta_S|^2 - \frac{m_1^2}{4})^2 + m_1^2 \rho_S^2]} \bar{A}_S(\rho_S, z_S), \\ A_z(\rho, z) &= -\frac{m_1^2 \rho_S^2 z_S}{2 [(|\zeta_S|^2 - \frac{m_1^2}{4})^2 + m_1^2 \rho_S^2]} \bar{B}_S(\rho_S, z_S) \\ &\quad + \left[\frac{|\zeta_S|^2}{|\zeta_S|^2 + \frac{m_1^2}{4}} - \frac{m_1^2 z_S^2 (|\zeta_S|^2 - \frac{m_1^2}{4})}{2(|\zeta_S|^2 + \frac{m_1^2}{4}) [(|\zeta_S|^2 - \frac{m_1^2}{4})^2 + m_1^2 \rho_S^2]} \right] \bar{A}_S(\rho_S, z_S). \end{aligned}$$

We thus write

$$\begin{aligned} B_\rho(\rho, z) &= \frac{|\zeta_S|^4}{2(|\zeta_S|^2 - \frac{m_1^2}{4})^2 [(|\zeta_S|^2 - \frac{m_1^2}{4})^2 + m_1^2 \rho_S^2]} \tilde{B}_S(\rho_S, z_S) \\ &= \frac{1}{\rho^2 + z^2 - m_1^2 + \sqrt{(\rho^2 + z^2 - m_1^2)^2 + 4m_1^2 \rho^2}} \times \\ &\quad \times \frac{1}{\sqrt{(\rho^2 + z^2 - m_1^2)^2 + 4m_1^2 \rho^2}} \tilde{B}_S(\rho_S, z_S), \quad (2.39) \end{aligned}$$

$$\begin{aligned} A_z(\rho, z) &= \frac{|\zeta_S|^2}{(|\zeta_S|^2 - \frac{m_1^2}{4})^2 + m_1^2 \rho_S^2} \tilde{A}_S(\rho_S, z_S) \\ &= \frac{1}{\sqrt{(\rho^2 + z^2 - m_1^2)^2 + 4m_1^2 \rho^2}} \tilde{A}_S(\rho_S, z_S), \quad (2.40) \end{aligned}$$

where $\tilde{B}_S, \tilde{A}_S \in C^\infty(\mathbb{C}_S \setminus D(0, \frac{m_1}{2}))$.

Finally, by (2.16) and (2.26) and the above regularity justification, we have

$$\begin{aligned} U &= o_{k-3}(r^{-1/2}), & \alpha &= o_{k-4}(r^{-1/2}), \\ B_\rho &= o_{k-3}(r^{-5/2}), & A_z &= o_{k-3}(r^{-3/2}). \end{aligned} \quad (2.41)$$

We have thus shown:

THEOREM 2.4 *Let (M, g) be a three-dimensional smooth simply connected manifold with a smooth connected compact boundary ∂M and assume that (M, g) admits a Killing vector field with periodic orbits. Furthermore, assume that (M, g) has one asymptotically flat end where it satisfies (2.1) for some $k \geq 5$. Then there exists a unique $m_1 > 0$ such that M is diffeomorphic to $\mathbb{R}^3 \setminus I$ for some line interval I of length $2m_1$, and, in cylindrical coordinates (ρ, z, φ) of \mathbb{R}^3 aligning so that $I = [-m_1, m_1]$ lies on the z -axis, the metric g takes the form (2.27), ∂_φ is the rotational Killing vector field of M , and U, α, A_ρ and B_z satisfy (2.29), (2.32), (2.33), (2.39), (2.40) and (2.41).*

REMARK 2.5 The above analysis can be carried out with some additional work to take care of the case where ∂M is disconnected. The only delicate point is the construction of the coordinates (ρ_S, z_S) such that, in the (ρ_S, z_S) -plane, ∂M corresponds to a union of a finite number of disjoint circles. An alternative way is to first construct the (ρ, z) coordinates as in [7, Section 6.3], and use our analysis here to derive the behaviour near each component of ∂M of the functions of interest. This approach simplifies the analysis in [7, Section 6.5].

2.4 The constant m_1

We showed earlier that m_1 is uniquely determined by the geometry of (M, g) . Here we give a more explicit description of m_1 .

Recall that $Q^{(2)}$ is represented by $\Omega_\#$ in $(\rho_\#, z_\#)$ -coordinates and that m_1 can be expressed in terms of the Robin constant $\gamma(\partial\Omega_\#)$ of $\partial\Omega_\#$ by (2.10). By definition, if $\Gamma = \Gamma_{\partial\Omega_\#}$ is the unique harmonic function in $\mathbb{C}_\#$ (with the flat metric) which vanishes at $\partial\Omega_\#$ and is asymptotic to $\frac{1}{2}\log(\rho_\#^2 + z_\#^2)$ at infinity, then

$$\Gamma(\rho_\#, z_\#) = \frac{1}{2}\log(\rho_\#^2 + z_\#^2) + \gamma(\partial\Omega_\#) + O((\rho_\#^2 + z_\#^2)^{-1/2}).$$

Let $\{y^1, y^2, y^3 = \varphi\}$ be a coordinate system on M such that $\{y^1, y^2\}$ is a coordinate system on $Q^{(2)}$. In the sequel, indices a and b range over $\{1, 2\}$, while Greek indices range over $\{1, 2, 3\}$. The induced quotient metric on $Q^{(2)}$ is given by

$$q_{ab} = g_{ab} - {}^K g^{\varphi\varphi} g_{\varphi a} g_{\varphi b} ,$$

where ${}^K g^{\varphi\varphi} = \frac{1}{g_{\varphi\varphi}}$. Note that, by (2.6),

$$q = e^{-2U_{\sharp} + 2\alpha_{\sharp}} (d\rho_{\sharp}^2 + dz_{\sharp}^2) .$$

Thus, as a function on $Q^{(2)}$, Γ is harmonic with respect to the metric q , i.e.

$$\partial_{y^a} \left(\sqrt{\det q} q^{ab} \partial_{y^b} \Gamma \right) = 0 \text{ in } \mathring{Q}^{(2)} .$$

Since Γ is φ -independent and ∂_{φ} is Killing, this implies that as a function on M , Γ satisfies

$$\partial_{y^{\mu}} \left(\sqrt{\frac{\det g}{g_{\varphi\varphi}}} g^{\mu\nu} \partial_{y^{\nu}} \Gamma \right) = 0 \text{ in } \mathring{M} \setminus \mathcal{A} . \quad (2.42)$$

We thus conclude that Γ satisfies

$$\begin{cases} L\Gamma := \Delta_g \Gamma - \frac{1}{2} g(\nabla_g \log g_{\varphi\varphi}, \nabla_g \Gamma) = 0 & \text{in } \mathring{M} \setminus \mathcal{A} , \\ \Gamma = 0 & \text{on } \partial M , \\ \Gamma = \log r + O(1) & \text{as } r \rightarrow \infty , \end{cases} \quad (2.43)$$

where r is the coordinate radius in the asymptotic region. Moreover, by construction, Γ is the unique solution to (2.43) satisfying $\partial_{\varphi} \Gamma \equiv 0$.

We thus have:

PROPOSITION 2.6 *The constant m_1 is given by*

$$m_1 = 2 \exp \left(- \lim_{r \rightarrow \infty} (\Gamma - \log r) \right) , \quad (2.44)$$

where Γ is the unique axially symmetric smooth solution to (2.43).

3 The ADM mass

In this section, we compute the ADM mass m of g as a volume integral over $\mathbb{R}^3 \setminus B(0, \frac{m_1}{2})$ and then use it to prove Theorem 1.1. We have

$$m = \lim_{R \rightarrow \infty} \frac{1}{16\pi} \int_{S_R} (g_{ij,j} - g_{jj,i}) \nu_i d\sigma ,$$

where the metric components are computed in a coordinate system satisfying (2.1)-(2.3), $d\sigma$ is the surface area form on S_R , and S_R can be taken to be any piecewise differentiable surface homologous to a coordinate sphere of radius R with

$$\inf\{r(p) : p \in S_R\} \rightarrow_{R \rightarrow \infty} \infty.$$

That the ADM mass is well-defined is well-known, see [2, 5].

3.1 Mass in pseudo-spherical coordinates

Define

$$x^1 = x_S = \rho_S \cos \varphi, \quad x^2 = y_S = \rho_S \sin \varphi, \quad x^3 = z_S.$$

Using (2.16), we can write the metric (2.15) as

$$\begin{aligned} g = & e^{-2U_S} (dx_S^2 + dy_S^2) + \frac{e^{-2U_S} (e^{2\alpha_S} - 1)}{\rho_S^2} (x_S dx_S + y_S dy_S)^2 \\ & + e^{-2U_S + 2\alpha_S} dz_S^2 + 2(x_S dy_S - y_S dx_S) (\bar{B}_S (x_S dx_S + y_S dy_S) \\ & + \bar{A}_S dz_S) + o_1(r^{-1}). \end{aligned} \quad (3.1)$$

Here r denotes the coordinate radius, $r = \sqrt{x_S^2 + y_S^2 + z_S^2}$.

In the following computation, S_R is the sphere of coordinate radius $r := \sqrt{x_S^2 + y_S^2 + z_S^2} = R$. Obviously, the error terms in (3.1) has no contribution to the mass integral. A straightforward computation using (2.15) shows that the terms involving \bar{B}_S and \bar{A}_S give also zero contribution to the mass integral.

The rest of the mass integrand is then found to be

$$\begin{aligned} & \left\{ \partial_{y_S} \left(e^{-2U_S} (e^{2\alpha_S} - 1) \frac{x_S y_S}{\rho_S^2} \right) \right. \\ & \quad \left. - \partial_{x_S} \left(e^{-2U_S} + e^{-2U_S} (e^{2\alpha_S} - 1) \frac{y_S^2}{\rho_S^2} \right) - \partial_{x_S} e^{-2U_S + 2\alpha_S} \right\} \frac{x_S}{r} \\ & + \left\{ \partial_{x_S} \left(e^{-2U_S} (e^{2\alpha_S} - 1) \frac{x_S y_S}{\rho_S^2} \right) \right. \\ & \quad \left. - \partial_{y_S} \left(e^{-2U_S} + e^{-2U_S} (e^{2\alpha_S} - 1) \frac{x_S^2}{\rho_S^2} \right) - \partial_{y_S} e^{-2U_S + 2\alpha_S} \right\} \frac{y_S}{r} \\ & + \left\{ -\partial_{z_S} \left(e^{-2U_S} + e^{-2U_S} (e^{2\alpha_S} - 1) \frac{x_S^2}{\rho_S^2} \right) \right. \\ & \quad \left. - \partial_{z_S} \left(e^{-2U_S} + e^{-2U_S} (e^{2\alpha_S} - 1) \frac{y_S^2}{\rho_S^2} \right) \right\} \frac{z_S}{r}. \end{aligned}$$

Upon simplifying this gives

$$\begin{aligned}
& \left\{ \partial_{y_S} \left(e^{-2U_S} (e^{2\alpha_S} - 1) \frac{x_S y_S}{\rho_S^2} \right) \right. \\
& \quad \left. - \partial_{x_S} \left(e^{-2U_S} \frac{x_S^2}{\rho_S^2} + e^{-2U_S+2\alpha_S} \left(1 + \frac{y_S^2}{\rho_S^2} \right) \right) \right\} \frac{x_S}{r} \\
& + \left\{ \partial_{x_S} \left(e^{-2U_S} (e^{2\alpha_S} - 1) \frac{x_S y_S}{\rho_S^2} \right) \right. \\
& \quad \left. - \partial_{y_S} \left(e^{-2U_S} \frac{y_S^2}{\rho_S^2} + e^{-2U_S+2\alpha_S} \left(1 + \frac{x_S^2}{\rho_S^2} \right) \right) \right\} \frac{y_S}{r} \\
& - \partial_{z_S} \left(e^{-2U_S} (e^{2\alpha_S} + 1) \right) \frac{z_S}{r} .
\end{aligned}$$

Expanding using (2.16) we obtain

$$\partial_R(2U_S - \alpha_S) + \frac{2}{r} \alpha_S + o(r^{-2}) .$$

We thus arrive at

$$m = \frac{1}{4\pi} \lim_{R \rightarrow \infty} \left\{ \int_{S_R} \partial_r \left(U_S - \frac{1}{2} \alpha_S \right) d\sigma + \frac{1}{2R} \int_{S_R} \alpha_S d\sigma \right\} . \quad (3.2)$$

(This is similar to a formula derived in [6], but the integrations are over different sets, which requires the new derivation above. The current expression is more convenient for our purposes.)

To proceed, we recall a formula for the scalar curvature on M from [9],

$$\begin{aligned}
R_g = 4 e^{2U_S-2\alpha_S} & \left[\Delta \left(U_S - \frac{1}{2} \alpha_S \right) - \frac{1}{2} |\nabla U_S|^2 + \frac{1}{2\rho_S} \partial_{\rho_S} \alpha_S \right. \\
& \left. - \frac{1}{8} \rho_S^2 e^{-2\alpha_S} (\rho_S \partial_{z_S} \bar{B}_S - \partial_{\rho_S} \bar{A}_S)^2 \right] . \quad (3.3)
\end{aligned}$$

Here Δ and ∇ are the Laplacian and the gradient operator taken with respect to the flat metric in \mathbb{R}^3 .

Using (3.3), we can convert (3.2) into volume integral form. Note that if Φ is a function defined on $\mathbb{R}^3 \setminus B(0, \frac{m_1}{2})$ satisfying

$$\Phi > 0 \text{ in } \mathbb{R}^3 \setminus B(0, \frac{m_1}{2}) , \text{ and } \Phi = 1 + o(r^{-1/2}) \text{ as } r \rightarrow \infty , \quad (3.4)$$

then

$$\lim_{R \rightarrow \infty} \int_{S_R} \partial_r \left(U_S - \frac{1}{2} \alpha_S \right) d\sigma = \lim_{R \rightarrow \infty} \int_{S_R} \Phi \partial_r \left(U_S - \frac{1}{2} \alpha_S \right) d\sigma . \quad (3.5)$$

By the divergence theorem, we have

$$\begin{aligned} & \int_{S_R} \Phi \partial_r \left(U_S - \frac{1}{2} \alpha_S \right) d\sigma \\ &= \int_{B(0,R) \setminus B(0, \frac{m_1}{2})} \left[\nabla \Phi \cdot \nabla \left(U_S - \frac{1}{2} \alpha_S \right) + \Phi \Delta \left(U_S - \frac{1}{2} \alpha_S \right) \right] d^3x \\ & \quad + \int_{\partial B(0, \frac{m_1}{2})} \Phi \partial_r \left(U_S - \frac{1}{2} \alpha_S \right) d\sigma . \end{aligned}$$

Hence, by (3.3),

$$\begin{aligned} \int_{S_R} \Phi \partial_r \left(U_S - \frac{1}{2} \alpha_S \right) d\sigma &= \int_{B(0,R) \setminus B(0, \frac{m_1}{2})} \left\{ \nabla \Phi \cdot \nabla \left(U_S - \frac{1}{2} \alpha_S \right) \right. \\ & \quad + \frac{1}{2} \Phi |\nabla U_S|^2 - \frac{1}{2\rho_S} \partial_{\rho_S} \alpha_S \Phi \\ & \quad + \frac{1}{4} e^{-2U_S + 2\alpha_S} \Phi R_g \\ & \quad + \frac{1}{8} \rho_S^2 e^{-2\alpha_S} \Phi (\rho_S \partial_{z_S} \bar{B}_S - \partial_{\rho_S} \bar{A}_S)^2 \Big\} d^3x \\ & \quad + \int_{\partial B(0, \frac{m_1}{2})} \Phi \partial_r \left(U_S - \frac{1}{2} \alpha_S \right) d\sigma . \end{aligned} \quad (3.6)$$

To get rid of the terms involving gradients of α_S we choose Φ to satisfy

$$\frac{1}{\rho_S} \operatorname{div} (\rho_S \nabla \Phi) = \operatorname{div} (\nabla \Phi + \Phi \nabla \log \rho_S) = 0 \text{ in } \mathbb{R}^3 \setminus B(0, \frac{m_1}{2}) . \quad (3.7)$$

Note that if we view Φ as a function defined in $\mathbb{R}^4 \setminus B(0, \frac{m_1}{2})$ invariant under $SO(2)$ and assume that Φ is locally bounded, then

$$\Delta^{(4)} \Phi = 0 \text{ in } \mathbb{R}^4 \setminus B(0, \frac{m_1}{2}) , \quad (3.8)$$

In particular, this implies that $\frac{1}{\rho_S} \partial_{\rho_S} \Phi$ is locally bounded, and $\partial \Phi = O(r^{-3})$ for large r . Thus, as α_S vanishes wherever $\rho_S = 0$, an application of the

divergence theorem gives

$$\begin{aligned}
& \int_{B(0,R) \setminus B(0, \frac{m_1}{2})} \left[\nabla \Phi \cdot \nabla \alpha_S + \frac{1}{\rho_S} \partial_{\rho_S} \alpha_S \Phi \right] d^3 x \\
&= \int_{B(0,R) \setminus B(0, \frac{m_1}{2})} \nabla \alpha_S \cdot (\nabla \Phi + \Phi \nabla \log \rho_S) d^3 x \\
&= \int_{S_R} \alpha_S (\partial_r \log \rho_S + O(R^{-3})) d\sigma \\
&\quad - \int_{\partial B(0, \frac{m_1}{2})} \alpha_S (\partial_r \Phi + \Phi \partial_r \log \rho_S) d\sigma \\
&= \frac{1}{R} \int_{S_R} \alpha_S d\sigma + o(R^{-3/2}) - \int_{\partial B(0, \frac{m_1}{2})} \alpha_S \left(\partial_r \Phi + \frac{2}{m_1} \Phi \right) d\sigma .
\end{aligned}$$

Substituting the above into (3.6) yields

$$\begin{aligned}
& \int_{S_R} \Phi \partial_r \left(U_S - \frac{1}{2} \alpha_S \right) d\sigma \\
&= \int_{B(0,R) \setminus B(0, \frac{m_1}{2})} \left\{ \nabla \Phi \cdot \nabla U_S + \frac{1}{2} \Phi |\nabla U_S|^2 \right. \\
&\quad + \frac{1}{4} e^{-2U_S + 2\alpha_S} \Phi R_g \\
&\quad + \frac{1}{8} \rho_S^2 e^{-2\alpha_S} \Phi (\rho_S \partial_{z_S} \bar{B}_S - \partial_{\rho_S} \bar{A}_S)^2 \Big\} d^3 x \\
&\quad - \frac{1}{2R} \int_{S_R} \alpha_S d\sigma + o(R^{-3/2}) \\
&\quad + \int_{\partial B(0, \frac{m_1}{2})} \left\{ \Phi \partial_r \left(U_S - \frac{1}{2} \alpha_S \right) + \frac{1}{2} \alpha_S \left(\partial_r \Phi + \frac{2}{m_1} \Phi \right) \right\} d\sigma .
\end{aligned}$$

Recalling (3.2) and (3.5), we arrive at

$$\begin{aligned}
m &= \frac{1}{4\pi} \int_{\mathbb{R}^3 \setminus B(0, \frac{m_1}{2})} \left\{ \nabla \Phi \cdot \nabla U_S + \frac{1}{2} \Phi |\nabla U_S|^2 \right. \\
&\quad + \frac{1}{4} e^{-2U_S + 2\alpha_S} \Phi R_g \\
&\quad + \frac{1}{8} \rho_S^2 e^{-2\alpha_S} \Phi (\rho_S \partial_{z_S} \bar{B}_S - \partial_{\rho_S} \bar{A}_S)^2 \Big\} d^3 x \\
&\quad + \frac{1}{4\pi} \int_{\partial B(0, \frac{m_1}{2})} \left\{ \Phi \partial_r \left(U_S - \frac{1}{2} \alpha_S \right) + \frac{1}{2} \alpha_S \left(\partial_r \Phi + \frac{2}{m_1} \Phi \right) \right\} d\sigma . \quad (3.9)
\end{aligned}$$

Next, if Ψ is a function defined on $\mathbb{R}^3 \setminus B(0, \frac{m_1}{3})$ such that

$$\begin{cases} \Delta \Psi = 0 & \text{in } \mathbb{R}^3 \setminus \bar{B}(0, \frac{m_1}{3}) , \\ \Psi = \text{Const} + O(r^{-1}) & \text{as } r \rightarrow \infty , \end{cases} \quad (3.10)$$

then

$$\int_{\mathbb{R}^3 \setminus B(0, \frac{m_1}{2})} \nabla \Psi \cdot \nabla U_S \, d^3x = - \int_{\partial B(0, \frac{m_1}{2})} U_S \partial_r \Psi \, d\sigma . \quad (3.11)$$

Using the above identity in (3.9) yields

$$\begin{aligned} m = & \frac{1}{4\pi} \int_{\mathbb{R}^3 \setminus B(0, \frac{m_1}{2})} \left\{ \frac{1}{2} \Phi |\nabla U_S|^2 + \nabla U_S \cdot \nabla (\Phi - \Psi) \right. \\ & + \frac{1}{4} e^{-2U_S + 2\alpha_S} \Phi R_g \\ & + \frac{1}{8} \rho_S^2 e^{-2\alpha_S} \Phi (\rho_S \partial_{z_S} \bar{B}_S - \partial_{\rho_S} \bar{A}_S)^2 \Big\} d^3x \\ & + \frac{1}{4\pi} \int_{\partial B(0, \frac{m_1}{2})} \left\{ \Phi \partial_r \left(U_S - \frac{1}{2} \alpha_S \right) + \frac{1}{2} \alpha_S \left(\partial_r \Phi + \frac{2}{m_1} \Phi \right) \right. \\ & \left. \left. - U_S \partial_r \Psi \right\} d\sigma . \end{aligned} \quad (3.12)$$

To conclude, we have shown:

PROPOSITION 3.1 *Under the hypotheses of Theorem 2.2 and (2.2), the ADM mass of (M, g) is well-defined and satisfies (3.12) for any Φ and Ψ satisfying (3.4), (3.7) and (3.10).*

We shall show below how appropriate choices of Φ and Ψ allow one to control the mass.

For further reference we note:

COROLLARY 3.2 *If (Φ_1, Ψ_1) and (Φ_2, Ψ_2) satisfy (3.4), (3.7) and (3.10) then*

$$\begin{aligned}
0 = & \frac{1}{4\pi} \int_{\mathbb{R}^3 \setminus B(0, \frac{m_1}{2})} \left\{ \frac{1}{2} (\Phi_1 - \Phi_2) |\nabla U_S|^2 + \nabla U_S \cdot \nabla (\Phi_1 - \Phi_2 - \Psi_1 + \Psi_2) \right. \\
& + \frac{1}{4} e^{-2U_S + 2\alpha_S} (\Phi_1 - \Phi_2) R_g \\
& + \frac{1}{8} \rho_S^2 e^{-2\alpha_S} (\Phi_1 - \Phi_2) (\rho_S \partial_{z_S} \bar{B}_S - \partial_{\rho_S} \bar{A}_S)^2 \Big\} d^3x \\
& + \frac{1}{4\pi} \int_{\partial B(0, \frac{m_1}{2})} \left\{ (\Phi_1 - \Phi_2) \partial_r \left(U_S - \frac{1}{2} \alpha_S \right) \right. \\
& + \frac{1}{2} \alpha_S \left(\partial_r \Phi_1 + \frac{2}{m_1} \Phi_1 - \partial_r \Phi_2 - \frac{2}{m_1} \Phi_2 \right) \\
& \left. - U_S (\partial_r \Psi_1 - \partial_r \Psi_2) \right\} d\sigma . \tag{3.13}
\end{aligned}$$

3.2 Lower bound for the ADM mass

In this section, we prove Theorem 1.1. We now assume that

$$R_g \geq 0 \text{ in } M , \tag{3.14}$$

together with a Riemannian version of the condition that ∂M is weakly outer trapped, namely:

$$\text{the mean curvature of } \partial M \text{ is non-positive} . \tag{3.15}$$

Here the mean curvature is computed with respect to the normal pointing towards M . By a direct computation, (3.15) is equivalent to

$$\partial_r \left(U_S - \frac{1}{2} \alpha_S \right) \geq \frac{2}{m_1} \text{ on } \partial B(0, \frac{m_1}{2}) . \tag{3.16}$$

PROOF OF THEOREM 1.1: Under (3.14) and (3.15), (3.12) implies, keeping in mind that Φ is positive, and completing the square in the volume integral when passing from the first to the second inequality,

$$\begin{aligned}
m \geq & \frac{1}{4\pi} \int_{\mathbb{R}^3 \setminus B(0, \frac{m_1}{2})} \left\{ \frac{1}{2} \Phi |\nabla U_S|^2 + \nabla U_S \cdot \nabla (\Phi - \Psi) \right\} d^3x \\
& + \frac{1}{4\pi} \int_{\partial B(0, \frac{m_1}{2})} \left\{ \Phi \frac{2}{m_1} + \frac{1}{2} \alpha_S \left(\partial_r \Phi + \frac{2}{m_1} \Phi \right) - U_S \partial_r \Psi \right\} d\sigma \\
\geq & -\frac{1}{8\pi} \int_{\mathbb{R}^3 \setminus B(0, \frac{m_1}{2})} \frac{1}{\Phi} |\nabla (\Phi - \Psi)|^2 d^3x \\
& + \frac{1}{4\pi} \int_{\partial B(0, \frac{m_1}{2})} \left\{ \Phi \frac{2}{m_1} + \frac{1}{2} \alpha_S \left(\partial_r \Phi + \frac{2}{m_1} \Phi \right) - U_S \partial_r \Psi \right\} d\sigma . \tag{3.17}
\end{aligned}$$

To continue, we specialize the choice of Φ and Ψ by taking

$$\Psi \equiv 1 \text{ and } \Phi \equiv 1 + \frac{m_1^2}{4r^2} .$$

Then (3.17) gives

$$\begin{aligned} m &\geq m_1 - \frac{1}{8\pi} \int_{\mathbb{R}^3 \setminus B(0, \frac{m_1}{2})} \frac{m_1^4}{r^4(4r^2 + m_1^2)} d^3x \\ &= \frac{\pi}{4} m_1 . \end{aligned} \quad (3.18)$$

Next, assume that $m = \frac{\pi}{4} m_1$. Then, we must have

$$R_g \equiv \rho_S \partial_{z_S} \bar{B}_S - \partial_{\rho_S} \bar{A}_S \equiv 0 \text{ in } \mathbb{R}^3 \setminus B(0, \frac{m_1}{2}) , \quad (3.19)$$

$$\nabla U_S \equiv -\frac{1}{\Phi} \nabla(\Phi - \Psi) = -\frac{1}{\Phi} \nabla \Phi \text{ in } \mathbb{R}^3 \setminus B(0, \frac{m_1}{2}) , \quad (3.20)$$

$$\partial_r \left(U_S - \frac{1}{2} \alpha_S \right) \equiv \frac{2}{m_1} \text{ on } \partial B(0, \frac{m_1}{2}) . \quad (3.21)$$

By (2.16), the second relation implies that

$$\nabla U_S \equiv \frac{2m_1^2}{r(4r^2 + m_1^2)} \partial_r \text{ in } \mathbb{R}^3 \setminus B(0, \frac{m_1}{2}) ,$$

and so, since U_S is assumed to asymptote to zero at infinity,

$$U_S \equiv \log \frac{4r^2}{4r^2 + m_1^2} \text{ in } \mathbb{R}^3 \setminus B(0, \frac{m_1}{2}) . \quad (3.22)$$

Taking (3.3), (3.21) and (3.22) into account we get

$$\begin{cases} \Delta \alpha_S - \frac{1}{\rho_S} \partial_{\rho_S} \alpha_S = -\frac{16m_1^2}{(4r^2 + m_1^2)^2} < 0 & \text{in } \mathbb{R}^3 \setminus \bar{B}(0, \frac{m_1}{2}) , \\ \partial_r \alpha_S = 0 & \text{on } \partial B(0, \frac{m_1}{2}) , \\ \alpha_S = o(r^{-1/2}) & \text{as } r \rightarrow \infty . \end{cases}$$

Since α_S is φ -independent, this implies

$$\begin{cases} \partial_{\rho_S}^2 \alpha_S + \partial_{z_S}^2 \alpha_S < 0 & \text{in } \mathbb{R}^2 \setminus \bar{D}(0, \frac{m_1}{2}) , \\ \partial_r \alpha_S = 0 & \text{on } \partial D(0, \frac{m_1}{2}) , \\ \alpha_S = o(r^{-1/2}) & \text{as } r \rightarrow \infty . \end{cases}$$

This is impossible by Hadamard's Three-Circle Theorem, proving that the equality cannot hold in (3.18). We conclude the proof of Theorem 1.1. \square

A A remark on the axisymmetric Penrose inequality

In [9] a proof of the Penrose inequality for axisymmetric initial data sets with positive scalar curvature has been given, under however undesirably stringent conditions on the geometry near the horizon. It seems therefore of interest to attempt to remove the overly restrictive conditions. In particular one can enquire whether our arguments above can be adapted to obtain the Penrose inequality. In this appendix we provide an argument that gives a result stronger than that in [9], but fails to provide the full Penrose inequality.

We will always assume (3.14), i.e. $R_g \geq 0$ in M . Furthermore, we will assume that

$$\partial M \text{ is minimal, i.e. } \partial_r \left(U_S - \frac{1}{2} \alpha_S \right) = \frac{2}{m_1} \text{ on } \partial B(0, \frac{m_1}{2}). \quad (\text{A.1})$$

By the first inequality in (3.17), we have

$$\begin{aligned} m \geq & \frac{1}{4\pi} \int_{\mathbb{R}^3 \setminus B(0, \frac{m_1}{2})} \left\{ \frac{1}{2} \Phi |\nabla U_S|^2 + \nabla U_S \cdot \nabla (\Phi - \Psi) \right\} d^3x \\ & + \frac{1}{4\pi} \int_{\partial B(0, \frac{m_1}{2})} \left\{ \frac{2}{m_1} \Phi + \frac{1}{2} \alpha_S \left(\partial_r \Phi + \frac{2}{m_1} \Phi \right) - U_S \partial_r \Psi \right\} d\sigma \end{aligned} \quad (\text{A.2})$$

for any Φ and Ψ satisfying (3.4), (3.7) and (3.10). Moreover, this is an equality iff $R_g \equiv 0 \equiv \rho_S \partial_{z_S} \bar{B}_S - \partial_{\rho_S} \bar{A}_S$.

Let A be the area of ∂M . Then

$$A = \int_{\partial B(0, \frac{m_1}{2})} e^{\alpha_S - 2U_S} d\sigma. \quad (\text{A.3})$$

According to Bray, Huisken, and Ilmanen [3, 12] one has

$$m \geq \sqrt{\frac{A}{16\pi}}. \quad (\text{A.4})$$

Hence, under the stated hypotheses and that $\mathbb{R}^3 \setminus B(0, \frac{m_1}{2})$ with the metric (2.15) contains no compact minimal surfaces other than its boundary,

one would naively expect that it must hold that

$$\begin{aligned}
J_{\Phi, \Psi}(U_S, \alpha_S) &:= \frac{1}{4\pi} \int_{\mathbb{R}^3 \setminus B(0, \frac{m_1}{2})} \left\{ \frac{1}{2} \Phi |\nabla U_S + \frac{1}{\Phi} \nabla(\Phi - \Psi)|^2 - \frac{1}{2\Phi} |\nabla(\Phi - \Psi)|^2 \right\} d^3x \\
&+ \frac{1}{4\pi} \int_{\partial B(0, \frac{m_1}{2})} \left\{ \frac{2}{m_1} \Phi + \frac{1}{2} \alpha_S \left(\partial_r \Phi + \frac{2}{m_1} \Phi \right) - U_S \partial_r \Psi \right\} d\sigma \\
&- \sqrt{\frac{1}{16\pi} \int_{\partial B(0, \frac{m_1}{2})} e^{\alpha_S - 2U_S} d\sigma} \geq 0, \tag{A.5}
\end{aligned}$$

for some well-chosen Φ and Ψ . Moreover equality should only hold for the Schwarzschild solution.

For a fixed m_1 this is thus a variational inequality: if the infimum over U_S and α_S as described above of $J_{\Phi, \Psi}(U_S, \alpha_S)$ is zero, then the axisymmetric Riemannian Penrose inequality would follow.

A natural choice for Φ and Ψ is to use functions which make the first volume integrand in (A.5) vanish for the Schwarzschild solution:

$$\nabla U_{S, \text{Schw}} + \frac{1}{\Phi} \nabla(\Phi - \Psi) \equiv 0, \tag{A.6}$$

where $U_{S, \text{Schw}}$ is the “ U_S ” of the Schwarzschildian slice,

$$U_{S, \text{Schw}} = -2 \log \frac{2r + m_1}{2r}.$$

This leads to $\Phi = 1 + \frac{a m_1^2}{4r}$ and $\Psi = \frac{b m_1}{2r}$. (Here we have used the equations (3.8) and (3.10).) Entering this into (A.6), we obtain $a = -1$ and $b = -2$.

There is a special case where the expected inequality holds:

PROPOSITION A.1 *For any (U_S, α_S) satisfying the relevant hypotheses and*

$$U_S - \frac{1}{2} \alpha_S \equiv C_H \geq -2 \log 2 \text{ on } \partial B(0, \frac{m_1}{2}), \tag{A.7}$$

there holds

$$J_{\Phi_*, \Psi_*}(U_S, \alpha_S) \geq 0,$$

where $\Phi_ = 1 - \frac{m_1^2}{4r^2}$ and $\Psi_* = -\frac{m_1}{r}$. Moreover, equality holds if and only if the metric (2.15) is that of a Schwarzschildian slice.*

It should be noted that the existence of admissible data verifying (A.7) (other than the Schwarzschildian slice) is not clear. We also note that the requirement that ∂M be the outermost minimal surface is not necessary, but rather ∂M being merely weakly outer trapped is sufficient.

Proposition A.1 should be compared with a result in [9], where equality in (A.7) is assumed together with the supplementary requirement that A , as defined by (A.3), equals $16\pi m_1^2$.

PROOF: We will only sketch the proof. Using the explicit form of (Φ_*, Ψ_*) , one finds

$$\begin{aligned} J_{\Phi_*, \Psi_*}(U_S, \alpha_S) = & -m_1(2 \log 2 - 1) - m_1 C_H - \frac{1}{4} m_1 e^{-C_H} \\ & + \frac{1}{8\pi} \int_{\mathbb{R}^3 \setminus B(0, \frac{m_1}{2})} \Phi |V_S|^2 d^3x, \end{aligned} \quad (\text{A.8})$$

where

$$V_S = \nabla U_S + \frac{1}{\Phi_*} \nabla(\Phi_* - \Psi_*).$$

Next, set $\Xi = \frac{m_1^2}{2r^2}$. Applying Corollary (3.2) to $(\Phi_1, \Psi_1) = (\Phi_* + \Xi, 0)$ and $(\Phi_2, \Psi_2) = (\Phi_*, \Psi_*)$ and noting (3.14) and (3.16), we find

$$\begin{aligned} 0 = & \frac{1}{4\pi} \int_{\mathbb{R}^3 \setminus B(0, \frac{m_1}{2})} \left\{ \frac{1}{2} \Xi |\nabla U_S|^2 + \nabla U_S \cdot \nabla(\Xi + \Psi_*) \right. \\ & + \frac{1}{4} e^{-2U_S + 2\alpha_S} \Xi R_g \\ & + \frac{1}{8} \rho_S^2 e^{-2\alpha_S} \Xi (\rho_S \partial_{z_S} \bar{B}_S - \partial_{\rho_S} \bar{A}_S)^2 \Big\} d^3x \\ & + \frac{1}{4\pi} \int_{\partial B(0, \frac{m_1}{2})} \left\{ \Xi \partial_r \left(U_S - \frac{1}{2} \alpha_S \right) \right. \\ & + \frac{1}{2} \alpha_S \left(\partial_r \Xi + \frac{2}{m_1} \Xi \right) + U_S \partial_r \Psi_* \Big\} d\sigma \\ \geq & \frac{1}{4\pi} \int_{\mathbb{R}^3 \setminus B(0, \frac{m_1}{2})} \left\{ \frac{1}{2} \Xi |\nabla U_S|^2 + \nabla U_S \cdot \nabla(\Xi + \Psi_*) \right\} d^3x \\ & + \frac{1}{4\pi} \int_{\partial B(0, \frac{m_1}{2})} \left\{ \frac{2}{m_1} \Xi + \frac{1}{2} \alpha_S \left(\partial_r \Xi + \frac{2}{m_1} \Xi \right) + U_S \partial_r \Psi_* \right\} d\sigma \end{aligned}$$

Using the explicit expressions for Ξ and Ψ_* , we then get

$$\begin{aligned} -m_1 C_H &= \frac{1}{4\pi} \int_{\partial B(0, \frac{m_1}{2})} \left[-\frac{1}{2} \alpha_S (\partial_r \Xi + \frac{2}{m_1} \Xi) - U_S \partial_r \Psi_* \right] d\sigma \\ &\geq \frac{1}{4\pi} \int_{\partial B(0, \frac{m_1}{2})} \frac{2}{m_1} \Xi d\sigma \\ &\quad + \frac{1}{4\pi} \int_{\mathbb{R}^3 \setminus B(0, \frac{m_1}{2})} \left\{ \nabla U_S \cdot \nabla (\Xi + \Psi_*) + \frac{1}{2} |\nabla U_S|^2 \Xi \right\} d^3x . \end{aligned}$$

Recalling that $\nabla U_S = V_S - \frac{1}{\Phi_*} \nabla (\Phi_* - \Psi_*)$, we thus have

$$\begin{aligned} -m_1 C_H &\geq \frac{1}{4\pi} \int_{\partial B(0, \frac{m_1}{2})} \frac{2}{m_1} \Xi d\sigma \\ &\quad + \frac{1}{4\pi} \int_{\mathbb{R}^3 \setminus B(0, \frac{m_1}{2})} \left\{ \frac{\Xi}{2\Phi_*^2} |\nabla (\Phi_* - \Psi_*)|^2 - \frac{1}{\Phi_*} \nabla (\Phi_* - \Psi_*) \cdot \nabla (\Xi + \Psi_*) \right\} d^3x \\ &\quad + \frac{1}{4\pi} \int_{\mathbb{R}^3 \setminus B(0, \frac{m_1}{2})} \left\{ V_S \cdot \left[\nabla (\Xi + \Psi_*) - \frac{\Xi}{\Phi_*} \nabla (\Phi_* - \Psi_*) \right] + \frac{1}{2} |V_S|^2 \Xi \right\} d^3x . \end{aligned}$$

Using the explicit expressions for Φ_* , Ψ_* and Ξ again, one arrives at

$$-m_1 C_H \geq 2 m_1 \log 2 - \sqrt{\frac{m_1}{8\pi}} \int_{\mathbb{R}^3 \setminus B(0, \frac{m_1}{2})} \Phi |V_S|^2 d^3x . \quad (\text{A.9})$$

Define

$$t = -C_H - 2 \log 2, \text{ and } l := \frac{1}{8\pi m_1} \int_{\mathbb{R}^3 \setminus B(0, \frac{m_1}{2})} \Phi |V_S|^2 d^3x$$

Then, by (A.8) and (A.9) and as $l \geq 0$ and $t \leq 0$ (by (A.7)),

$$\begin{aligned} \frac{1}{m_1} J_{\Phi_*, \Psi_*}(U_S, \alpha_S) &= 1 + t - e^t + l \\ &\geq 1 - \sqrt{l} - e^{-\sqrt{l}} + l \geq 0 . \end{aligned}$$

This finishes the proof. \square

The bad news for the above program arises from the following:

PROPOSITION A.2 *There exists an “admissible” $(U_S, \alpha_S) = (U_S, 0)$ such that*

1. $U_S - \frac{1}{2}\alpha_S \equiv C_H < -2\log 2$ on $\partial B(0, \frac{m_1}{2})$,

2. $J_{\Phi_*, \Psi_*}(U_S, \alpha_S) < 0$.

PROOF: The example is provided by conformally Schwarzschildian metrics, i.e. $\alpha_S \equiv 0$. U_S is given by

$$U_S = U_S^{(k)} = -2\log \left[1 + \frac{6k + m_1^2}{2m_1 r} - \frac{k}{2r^2} \right], \quad k \geq 0.$$

For $k = 0$, this gives exactly the Schwarzschild metric. For $k > 0$, the scalar curvature is readily seen to be positive, as $e^{-U_S/2}$ is super-harmonic (with respect to the flat metric). One can check directly from (A.1) that ∂M is minimal. In fact, for $k < \frac{m_1^2}{6}$, ∂M is outermost minimal. (An easy way to see that is to check that, for those values of k , the coordinate spheres provide a foliations of M by constant positive mean curvature surfaces.) The rest of the argument is to use (A.8) to verify that $J_{\Phi_*, \Psi_*}(U_S, \alpha_S)$ is negative for sufficiently small $k > 0$. \square

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